Sum of squares of tangents and Chebishev Polynomials.
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Evaluate $\tan^2 \dfrac{\pi}{2022} + \tan^2 \dfrac{2\pi}{2022} + \tan^2 \dfrac{3\pi}{2022} + \ldots + \tan^2 \dfrac{1010\pi}{2022} = ?$.

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We will find $\displaystyle\sum_{k=1}^{n-1} \tan^2 \dfrac{k\pi}{2n}$, $n \geq 2$. Since $\displaystyle\sum_{k=1}^{n-1} \tan^2 \dfrac{k\pi}{2n} = \sum_{k=1}^{n-1} \left(\dfrac{1}{\cos^2 \frac{k\pi}{2n}} - 1 \right) =$

$\displaystyle\sum_{k=1}^{n-1} \dfrac{1}{\cos^2 \frac{k\pi}{2n}} - n + 1$ then our problem reduced to calculation of the sum $\displaystyle\sum_{k=1}^{n-1} \dfrac{1}{\cos^2 \frac{k\pi}{2n}}$.

First note, that for any polynomial $P(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1}x + a_n \ (a_0 \neq 0)$
with non-zero roots $x_1, x_2, \ldots, x_n$ holds

(1) $\displaystyle\sum_{i=1}^{n} \dfrac{1}{x_i} = -\dfrac{P'(0)}{P(0)}$.

Indeed, since $P(x) = a_0(x - x_1)(x - x_2)\ldots(x - x_n)$ then

$\displaystyle\sum_{i=1}^{n} \dfrac{1}{x - x_i} = \sum_{i=1}^{n} (\ln(x - x_i))' = \left(\sum_{i=1}^{n} \ln(x - x_i) \right)' = \left(\ln \dfrac{P(x)}{a_0} \right)' = \dfrac{P'(x)}{P(x)}$.

Hence, $\displaystyle\sum_{i=1}^{n} \dfrac{1}{x_i} = -\dfrac{P'(0)}{P(0)}$.

Let $U_n(x) := \dfrac{T'_{n+1}(x)}{n+1} = \dfrac{\sin(n+1)\varphi}{\sin\varphi}$ is Chebishev Polynomial of the Second Kind.

Then $U_n(x)$ satisfies to recurrence
$U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x), n \in \mathbb{N}$ and $U_0(x) = 1, U_1(x) = 2x$.

Since $\dfrac{\sin n\varphi}{\sin\varphi} = 0 \Leftrightarrow \varphi = \dfrac{k\pi}{n}, n \in \mathbb{Z}$ then $U_{n-1}(x) = 0 \Leftrightarrow x = \cos \dfrac{k\pi}{n}, k = 1, 2, \ldots, n-1$

and $U_{n-1}(x) = 2^{n-1} \left(x - \cos \dfrac{\pi}{n} \right)\left(x - \cos \dfrac{2\pi}{n} \right)\ldots\left(x - \cos \dfrac{(n-1)\pi}{n} \right)$.

(If α_n is coefficient for x^n in $U_n(x)$ then α_n satisfies to recurrence $\alpha_{n+1} = 2\alpha_n, \alpha_0 = 1$).

In particularly $U_{2n-1}(x) = 2^{2n-1} \displaystyle\prod_{k=1}^{2n-1} \left(x - \cos \dfrac{k\pi}{2n} \right) =$

$2^{2n-1}\left(x - \cos \dfrac{n\pi}{2n} \right)\displaystyle\prod_{k=1}^{n-1}\left(x - \cos \dfrac{k\pi}{2n} \right)\prod_{k=1}^{n-1}\left(x - \cos \dfrac{(2n-k)\pi}{2n} \right) = 2^{2n-1}x\prod_{k=1}^{n-1}\left(x^2 - \cos^2 \dfrac{k\pi}{2n} \right)$.

Let $P_n(x) := \dfrac{U_{2n-1}(\sqrt{x})}{2\sqrt{x}}$ then $P_n(x) = 4^{n-1}\displaystyle\prod_{k=1}^{n-1}\left(x - \cos^2 \dfrac{k\pi}{2n} \right)$.

Note that $U_{2n-1}(x)$ can be directly defined by recurrence
$U_{2n+1}(x) = 2(2x^2 - 1)U_{2n-1}(x) - U_{2n-3}(x), n \in \mathbb{N}$ with $U_{-1}(x) = 0, U_1(x) = 2x$.
Since $U_{2n-1}(x)$ divisible by $2x$ then polynomial $P_n(x)$ satisfy to the recurrence
(2) $P_{n+1}(x) = 2(2x - 1)P_n(x) - P_{n-1}(x), n \in \mathbb{N}$ with $P_0(x) = 0, P_1(x) = 1$.

Thus, applying **(1)** to polynomial $P_n(x)$ we obtain $\displaystyle\sum_{k=1}^{n-1} \dfrac{1}{\cos^2 \frac{k\pi}{2n}} = -\dfrac{P'_n(0)}{P_n(0)}$.

In particularly, from **(2)** follows the recurrence

(3) $P_{n+1}(0) + 2P_n(0) + P_{n-1}(0) = 0, n \in \mathbb{N}$ with $P_0(0) = 0, P_1(0) = 1$.

Let $b_n := \dfrac{P_n(0)}{(-1)^n}$ then **(3)** can be rewritten as

(4) $\quad b_{n+1} - 2b_n + b_{n-1} = 0, n \in \mathbb{N}$ with $b_0 = 0, b_1 = -1$.

Since $b_{n+1} - b_n = b_n - b_{n-1}, n \in \mathbb{N}$ then $b_n - b_{n-1} = -1, n \in \mathbb{N}$ and, therefore,

$$\sum_{k=1}^{n}(b_k - b_{k-1}) = -n \iff b_n - b_0 = -n \iff b_n = -n.$$

From the other hand, since $P'_{n+1}(x) = 2(2x - 1)P'_n(x) + 4P_n(x) - P'_{n-1}(x), n \in \mathbb{N}$

with $P'_0(x) = 0, P'_1(x) = 0$, then

(5) $\quad P'_{n+1}(0) + 2P'_n(0) + P'_{n-1}(0) = 4P_n(0), n \in \mathbb{N}$ with $P'_0(0) = 0, P'_1(0) = 0$.

Let $a_n := \dfrac{P'_n(0)}{(-1)^n}, n \in \mathbb{N} \cup \{0\}$ then $\dfrac{P_n(0)}{(-1)^{n+1}} = -b_n = n$ and **(5)** can be rewritten as

(6) $\quad a_{n+1} - 2a_n + a_{n-1} = 4n, n \in \mathbb{N}$ with $a_0 = a_1 = 0$.

Since sequence $\left(\dfrac{2n(n^2 - 1)}{3} \right)$ is particular solution of nonhomogeneous recurrence **(6)**

then $a_n = \dfrac{2n(n^2 - 1)}{3} + \alpha n + \beta$ where $\alpha = \beta = 0$ because $a_0 = a_1 = 0$.

Thus $a_n = \dfrac{2n(n^2 - 1)}{3}, n \in \mathbb{N}$ and

$$\sum_{k=1}^{n-1} \frac{1}{\cos^2 \frac{k\pi}{2n}} = -\frac{P'_n(0)}{P_n(0)} = \frac{\dfrac{P'_n(0)}{(-1)^n}}{\dfrac{P_n(0)}{(-1)^{n+1}}} = \frac{a_n}{n} = \frac{2(n^2 - 1)}{3}.$$

Hence, $\displaystyle\sum_{k=1}^{n-1} \tan^2 \frac{k\pi}{2n} = \frac{2(n^2 - 1)}{3} - n + 1 = \frac{(n-1)(2n-1)}{3}$.

In particular, for $n = 1011$ we obtain $\displaystyle\sum_{k=1}^{1010} \tan^2 \frac{k\pi}{2022} = \frac{(1011 - 1)(2022 - 1)}{3} = \frac{2041\,210}{3}$.